

# Resummation Approach in QCD Analytic Perturbation Theory

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## Abstract

We discuss the resummation approach in QCD Analytic Perturbation Theory (APT). We start with a simple example of asymptotic power series for a zero-dimensional analog of the scalar  $g\varphi^4$  model. Then we give a short historic preamble of APT and show that renormgroup improvement of the QCD perturbation theory dictates to use the Fractional APT (FAPT). After that we discuss the (F)APT resummation of nonpower series and provide the one-, two-, and three-loop resummation recipes. We show the results of applications of these recipes to the estimation of the Adler function  $D(Q^2)$  in the  $N_f = 4$  region of  $Q^2$  and of the Higgs-boson-decay width  $\Gamma_{H \rightarrow b\bar{b}}(m_H^2)$  for  $M_H = 100 - 180 \text{ GeV}^2$ .

**Keywords:** Renormalization group, QCD, Analytic Perturbation Theory, Nonpower Series Resummation, Adler function, Higgs boson decay

## 1. Simple example of asymptotic power series

In spite of many examples of successful applications of perturbative approach in quantum field theory, a perturbative power expansion for a quantum amplitude usually is not convergent. Instead, a typical power series appears to be asymptotic. To refresh the reader knowledge on this subject we consider a simple example — the so-called “0-dimensional” analog of the scalar field theory  $g\varphi^4$ :

$$I(g) = \int_{-\infty}^{\infty} e^{-x^2 - gx^4} dx. \quad (1a)$$

It can be expanded in a power series [1]

$$I(g) = \sum_{k=0}^{\infty} (-g)^k I_k, \quad (1b)$$

with factorially growing coefficients:  $I_k = \Gamma(2k + 1/2)/\Gamma(k + 1) \rightarrow 2^k k!$  for  $k \gg 1$ . Meanwhile,  $I(g)$  can be expressed via special MacDonald function

$$I(g) = \frac{1}{2\sqrt{g}} e^{1/8g} K_{1/4}\left(\frac{1}{8g}\right) \quad (1c)$$

with known analytic properties in the complex  $g$  plane: It is a four-sheeted function analytical in the whole complex plane besides the cut from the origin  $g = 0$  along the whole negative semiaxis. It has an essential singularity  $e^{-1/8g}$  at the origin and in its vicinity on the first Riemann sheet it can be written down in the Cauchy integral form:

$$I(g) = \sqrt{\pi} - \frac{g}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\gamma e^{-1/4\gamma}}{\gamma(g + \gamma)}. \quad (1d)$$

Due to this singular behavior near the origin the power Taylor series (1b) has no convergence domain for real positive  $g$  values. This behavior is in one-to-one correspondence with the factorial growth of power expansion coefficients. The same factorial growth of expansion coefficients has been proved for the  $\phi^4$  scalar and a few other QFT models [2].

Power series with factorially growing coefficients belongs to the class of Asymptotic Series (AS) — their properties were investigated by Henry Poincaré at the end of the XIX century. In short, he concluded that the truncated AS can be used for obtaining the quantitative information on expanded function. To be more concrete, the error of approximating  $F(g)$  by first  $K$  terms

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Table 1: The last detained  $((-g)^K I_K)$  and the first dismissed  $((-g)^{K+1} I_{K+1})$  contributions to the power series (1b) in comparison with the exact value (1c) and the approximate result  $I_K(g)$ , with  $K$  being the truncation number and  $\Delta_K I(g)$  — the error of approximation.

$g$	$K$	$(-g)^K I_K$	$(-g)^{K+1} I_{K+1}$	$I_K(g)$	$I(g)$	$\Delta_K I(g)$
0.07	<b>7</b>	-0.04(2%)	+0.07(4.4%)	1.674	1.698	<b>1.4%</b>
0.07	9	-0.17(10%)	+0.42(25%)	1.582	1.698	7%
0.15	<b>2</b>	+0.13(8%)	-0.16(10%)	1.704	1.639	<b>4%</b>
0.15	4	+0.30(18%)	-0.72(44%)	1.838	1.639	12%

of expansion,  $F(g) \rightarrow F_K(g) = \sum_{k \leq K} f_k(g)$ , is equal to the last accounted term  $f_K(g)$ . This observation can be used to obtain a lower limit of possible accuracy for the given  $g$  value: one should find the number  $K$  with the minimal value of  $f_K(g)$  — then its absolute value just set the limit of accuracy.

To make this statement more clear, we take a power AS,  $f_k(g) = f_k g^k$ , with factorial growth of expansion coefficients,  $f_k \sim k!$ . Then we see that terms  $f_k(g)$  stop to decay at  $k \sim K \sim 1/g$ , hence for  $k \gtrsim K + 1$  truncation error starts to grow. The explicit calculations for the function  $I(g)$  (1a) with AS (1b), presented in Table 1 (see also in [3]), shows that the optimal values of truncation number  $K = K_* \simeq 1/(2g)$  (in dark green) indeed provides the best possible accuracy (dark green in the last column): Attempting to account a couple of extra terms (the second and the forth lines) results in drastic rise of the error! Thus, one has  $K_*(g = 0.07) = 7$  and  $K_*(g = 0.15) = 2$ . It is not possible at all to get the 1% accuracy for  $g = 0.15$ , instead the best accuracy here is 4%. In this situation to obtain more accurate result we need to use Eq. (1c) which has the right analytic properties in  $g$ .

We conclude from this example that to produce reliable predictions for a physical amplitude in QCD one needs a tool for resumming the original perturbative power series into the quasi-nonperturbative amplitude which should have prescribed analytical properties, dictated by the first principles of QCD, like causality and renormalizability. It is intriguing that the APT approach in QCD allows one to obtain such a tool.

## 2. Analytic Perturbation Theory in QCD

In the standard QCD PT we know the Renormalization Group (RG) equation in the  $l$ -loop approximation

$$\frac{da_{(l)}[L]}{dL} = -a_{(l)}^2[L] \left[ 1 + \sum_{k \geq 1}^{l-1} c_k a_{(l)}^k[L] \right] \quad (2)$$

for the effective coupling  $\alpha_s(Q^2) = a_{(l)}[L]/\beta_f$  with  $L = \ln(Q^2/\Lambda^2)$ ,  $\beta_f = b_0(N_f)/(4\pi) = (11 - 2N_f/3)/(4\pi)$ .<sup>1</sup> Then its one-loop solution generates Landau pole singularity,  $a_{(1)}[L] = 1/L$ . As a consequence, the perturbative power series for the Adler function,  $D(Q^2) = d_0 + \sum_{k \geq 1} d_k a^k[L]$ , being reasonably good in the deep UV region of  $Q^2$ , at  $Q^2 = \Lambda^2$  has the unphysical pole — in marked contrast to the analyticity property of the Adler function. Indeed, it should have a cut along the negative axis of  $Q^2$ , but can not have any poles in Euclidean domain,  $Q^2 > 0$ . APT was suggested as a resolution of this contradiction: From the very beginning we demand that improved coupling, as well as all its integer powers, should have the right analytic properties, i. e. be represented as dispersive integrals, see (4).

Strictly speaking the QCD Analytic Perturbation Theory (APT) was initiated by N. N. Bogoliubov et al. paper of 1959 [4], where ghost-free effective coupling for QED has been constructed. Then in 1982 Radyushkin [5] and Krasnikov and Pivovarov [6] using the same dispersion technique suggested regular (for  $s \geq \Lambda^2$ ) QCD running coupling in Minkowskian region, the well-known  $\pi^{-1} \arctan(\pi/L)$ . After that in 1995 Jones and Solovtsov using variational approach [7] constructed the effective couplings in Euclidean and Minkowski domains which appears to be finite for all  $Q^2$  and  $s$  and satisfy analyticity integral conditions (3). Just in the same time Shirkov and Solovtsov [8], using the dispersion approach of [4], discovered ghost-free coupling  $\mathcal{A}_1[L]$ , Eq. (5a), in Euclidean region and ghost-free coupling  $\mathfrak{A}_1[L]$ , Eq. (5b), in Minkowskian region, which satisfy analyticity integral conditions

$$\mathcal{A}_1(Q^2) = \hat{D}[\mathfrak{A}_1](Q^2) \equiv Q^2 \int_0^\infty \frac{\mathfrak{A}_1(\sigma)}{(\sigma + Q^2)^2} d\sigma; \quad (3a)$$

$$\mathfrak{A}_1(s) = \hat{R}[\mathcal{A}_1](s) \equiv \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} \frac{\mathcal{A}_1(\sigma)}{\sigma} d\sigma. \quad (3b)$$

<sup>1</sup>We use notations  $f(Q^2)$  and  $f[L]$  in order to specify the arguments we mean — squared momentum  $Q^2$  or its logarithm  $L = \ln(Q^2/\Lambda^2)$ , that is  $f[L] = f(\Lambda^2 \cdot e^L)$  and the QCD scale parameter  $\Lambda$  is usually referred to  $N_f = 3$  region.

The last coupling coincides with the Radyushkin one for  $s \geq \Lambda^2$ . Due to the absence of singularities in these couplings, Shirkov and Solovtsov suggested to use them for all  $Q^2$  and  $s$  (for recent applications — see in [9]).

Shirkov–Solovtsov approach, now termed APT, appears to be very powerful: in Euclidean domain,  $-q^2 = Q^2$ ,  $L = \ln Q^2/\Lambda^2$ , it generates the following set of images for the effective coupling and its  $n$ -th powers,  $\{\mathcal{A}_n[L]\}_{n \in \mathbb{N}}$ , whereas in Minkowskian domain,  $q^2 = s$ ,  $L_s = \ln s/\Lambda^2$ , it generates another set,  $\{\mathfrak{A}_n[L_s]\}_{n \in \mathbb{N}}$ . APT is based on the RG and causality that guarantees standard perturbative UV asymptotics and spectral properties. Power series of the standard PT  $\sum_m d_m a^m[L]$  transforms into non-power series  $\sum_m d_m \mathcal{A}_m[L]$  in APT.

By the analytization in APT for an observable  $f(Q^2)$  we mean the dispersive “Källén–Lehman” representation

$$[f(Q^2)]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon} d\sigma \quad (4)$$

with  $\rho_f(\sigma) = \pi^{-1} \text{Im} [f(-\sigma)]$ . Then in the one-loop approximation  $\rho_1^{(1)}(\sigma) = 1/\sqrt{L_\sigma^2 + \pi^2}$  and

$$\mathcal{A}_1^{(1)}[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2} d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}, \quad (5a)$$

$$\mathfrak{A}_1^{(1)}[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma} d\sigma = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}, \quad (5b)$$

whereas analytic images of the higher powers ( $n \geq 2, n \in \mathbb{N}$ ) are:

$$\left( \frac{\mathcal{A}_n^{(1)}[L]}{\mathfrak{A}_n^{(1)}[L_s]} \right) = \frac{1}{(n-1)!} \left( -\frac{d}{dL} \right)^{n-1} \left( \frac{\mathcal{A}_1^{(1)}[L]}{\mathfrak{A}_1^{(1)}[L_s]} \right). \quad (6)$$

### 3. Fractional APT in QCD

At first glance, the APT is a complete theory providing tools to produce an analytic answer for any perturbative series in QCD. But in 2001 Karanikas and Stefanis [10] suggested the principle of analytization “as a whole” in the  $Q^2$  plane for hadronic observables, calculated perturbatively. More precisely, they proposed the analytization recipe for terms like  $\int_0^1 dx \int_0^1 dy \alpha_s(Q^2 xy) f(x) f(y)$ , which can be treated as an effective account for the logarithmic terms in the next-to-leading-order approximation of the perturbative QCD. This actually generalizes the analytic approach suggested in [11]. Indeed, in the standard QCD PT one has also:

- (i) the factorization QCD procedure that gives rise to the appearance of logarithmic factors of the type:  $a^\nu[L] L$ ;
- (ii) the RG evolution that generates evolution factors of

the type:  $B(Q^2) = [Z(Q^2)/Z(\mu^2)] B(\mu^2)$ , which reduce in the one-loop approximation to  $Z(Q^2) \sim a^\nu[L]$  with  $\nu = \gamma_0/(2b_0)$  being a fractional number.

All that means that in order to generalize APT in the “analytization as a whole” direction one needs to construct analytic images of new functions:  $a^\nu$ ,  $a^\nu L^m, \dots$ . This task has been performed in the frames of the so-called FAPT, suggested in [12, 13]. Now we briefly describe this approach.

In the one-loop approximation using recursive relation (6) we can obtain explicit expressions for both couplings:

$$\mathcal{A}_\nu^{(1)}[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}; \quad (7a)$$

$$\mathfrak{A}_\nu^{(1)}[L] = \frac{\sin \left[ (\nu-1) \arccos \left( \frac{L}{\sqrt{\pi^2 + L^2}} \right) \right]}{\pi(\nu-1)(\pi^2 + L^2)^{(\nu-1)/2}}. \quad (7b)$$

Here  $F(z, \nu)$  is reduced Lerch transcendental function, which is an analytic function in  $\nu$ . The obtained functions,  $\mathcal{A}_\nu^{(1)}[L]$  and  $\mathfrak{A}_\nu^{(1)}[L]$ , have very interesting properties, which we discussed extensively in our previous papers [12–15]. Note here, that in the one-loop approximation to find analytic images of  $a_{(1)}^\nu[L] \cdot L^m$  is very easy: they are just  $\mathcal{A}_{\nu-m}^{(1)}[L]$  and  $\mathfrak{A}_{\nu-m}^{(1)}[L]$ .

Constructing FAPT in the higher- $(l)$ -loop approximations is a more complicated task. Here we represent the original  $l$ -loop running coupling in the following form:

$$a_{(l)}[L_\sigma - i\pi] = \frac{e^{i\varphi_{(l)}[L_\sigma]}}{R_{(l)}[L_\sigma]}, \quad (8)$$

with  $R_{(l)}[L]$  and  $\varphi_{(l)}[L]$  being the known functions. Then the spectral densities of the  $\nu$ -power of the coupling is

$$\rho_\nu^{(l)}[L_\sigma] = \frac{1}{\pi} \frac{\sin[\nu \varphi_{(l)}[L_\sigma]]}{(R_{(l)}[L_\sigma])^\nu}, \quad (9)$$

but spectral densities of  $a^\nu[L] \cdot L^m$  start to be more complicated:

$$\rho_{L_{\nu,m}}^{(l)}[L] = \frac{R_{(1)}^m[L]}{\pi R_{(l)}^\nu[L]} \sin[\nu \varphi_{(l)}[L] - m \varphi_{(1)}[L]]. \quad (10)$$

Construction of FAPT with fixed number of quark flavors,  $N_f$ , is a two-step procedure: we start with the perturbative result  $[a(Q^2)]^\nu$ , generate the spectral density  $\rho_\nu(\sigma)$  using Eq. (4), and then obtain analytic couplings  $\mathcal{A}_\nu[L]$  and  $\mathfrak{A}_\nu[L]$  via Eqs. (5). Here  $N_f$  is fixed and factorized out. We can proceed in the same manner for  $N_f$ -dependent quantities:  $[\alpha_s(Q^2; N_f)]^\nu \Rightarrow \bar{\rho}_\nu(\sigma; N_f) = \bar{\rho}_\nu[L_\sigma; N_f] \equiv \rho_\nu(\sigma)/\beta_f^\nu \Rightarrow \bar{\mathcal{A}}_\nu[L; N_f]$  and  $\bar{\mathfrak{A}}_\nu[L; N_f]$  — here  $N_f$  is fixed, but not factorized out.

Global version of FAPT [14], which takes into account heavy-quark thresholds, is constructed along the same lines but starting from global perturbative coupling  $[\alpha_s^{\text{glob}}(Q^2)]^\nu$ , being a continuous function of  $Q^2$  due to choosing different values of QCD scales  $\Lambda_f$ , corresponding to different values of  $N_f$ . We illustrate here the case of only one heavy-quark threshold at  $s = m_4^2$ , corresponding to the transition  $N_f = 3 \rightarrow N_f = 4$ . Then we obtain the discontinuous spectral density

$$\rho_n^{\text{glob}}(\sigma) = \theta(L_\sigma < L_4) \bar{\rho}_n[L_\sigma; 3] + \theta(L_4 \leq L_\sigma) \bar{\rho}_n[L_\sigma + \lambda_4; 4], \quad (11)$$

with  $L_\sigma \equiv \ln(\sigma/\Lambda_3^2)$ ,  $L_f \equiv \ln(m_f^2/\Lambda_3^2)$  and  $\lambda_f \equiv \ln(\Lambda_3^2/\Lambda_f^2)$  for  $f = 4$ , which is expressed in terms of fixed-flavor spectral densities with 3 and 4 flavors,  $\bar{\rho}_n[L; 3]$  and  $\bar{\rho}_n[L + \lambda_4; 4]$ . However it generates the continuous Minkowskian coupling

$$\mathfrak{A}_\nu^{\text{glob}}[L] = \theta(L < L_4) (\bar{\mathfrak{A}}_\nu[L; 3] + \Delta_{43} \bar{\mathfrak{A}}_\nu) + \theta(L_4 \leq L) \bar{\mathfrak{A}}_\nu[L + \lambda_4; 4]. \quad (12a)$$

with  $\Delta_{43} \bar{\mathfrak{A}}_\nu = \bar{\mathfrak{A}}_\nu[L_4 + \lambda_4; 4] - \bar{\mathfrak{A}}_\nu[L_4; 3]$  and the analytic Euclidean coupling  $\mathcal{A}_\nu^{\text{glob}}[L]$

$$\mathcal{A}_\nu^{\text{glob}}[L] = \bar{\mathcal{A}}_\nu[L + \lambda_4; 4] + \int_{-\infty}^{L_4} \frac{\bar{\rho}_\nu[L_\sigma; 3] - \bar{\rho}_\nu[L_\sigma + \lambda_4; 4]}{1 + e^{L-L_\sigma}} dL_\sigma \quad (12b)$$

(for more detail see in [14]).

#### 4. Resummation in (F)APT

Before starting with the definite-loop approximation we introduce here, following [16], the generating function  $P(t)$  for perturbative coefficients  $d_k$  which allows us then to resum any non-power series ( $\nu = 0$  corresponds to the APT case) of the type

$$\mathcal{S}_\nu[L; \mathcal{F}] = d_0 \mathcal{F}_\nu[L] + d_1 \sum_{n \geq 1} \tilde{d}_n \mathcal{F}_{n+\nu}[L], \quad (13)$$

where  $\mathcal{F}[L]$  denotes one of the analytic quantities  $\mathcal{A}^{(l)}[L]$ ,  $\mathfrak{A}^{(l)}[L]$ , or  $\rho^{(l)}[L]$ , and  $\tilde{d}_n \equiv d_n/d_1$ . We suppose that

$$\tilde{d}_n = \int_0^\infty P(t) t^{n-1} dt \quad \text{with} \quad \int_0^\infty P(t) dt = 1. \quad (14)$$

To shorten our formulas we use the abbreviated notation

$$\langle\langle F[L, t] \rangle\rangle_{P(t)} \equiv \int_0^\infty F[L, t] P(t) dt. \quad (15)$$

Then  $\tilde{d}_n = \langle\langle t^{n-1} \rangle\rangle_{P(t)}$  and we need to resum the series

$$\mathcal{W}_\nu[L; t; \mathcal{F}] = \sum_{n \geq 1} t^{n-1} \mathcal{F}_{n+\nu}[L], \quad (16a)$$

related to the original one in a simple way

$$\mathcal{S}_\nu[L; \mathcal{F}] = d_0 \mathcal{F}_\nu[L] + d_1 \langle\langle \mathcal{W}_\nu[L; t; \mathcal{F}] \rangle\rangle_{P(t)}. \quad (16b)$$

##### 4.1. One-loop FAPT

In the one-loop approximation we have the following recurrence relation ( $\dot{\mathcal{F}}[L] \equiv d\mathcal{F}[L]/dL$ ):

$$-\frac{1}{n+\nu} \dot{\mathcal{F}}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L]. \quad (17)$$

This property of couplings and spectral densities allows us to resum the series (16b):

$$\mathcal{S}_\nu[L, \mathcal{F}] = d_0 \mathcal{F}_\nu[L] + \hat{d}_1 \langle\langle \mathcal{F}_{1+\nu}[L - t] \rangle\rangle_{P_\nu(t)}, \quad (18a)$$

where now the generating function  $P_\nu$  depends on  $\nu$ ,

$$P_\nu(t) = \int_0^1 P\left(\frac{t}{1-x}\right) \Phi_\nu(x) \frac{dx}{1-x}. \quad (18b)$$

Here  $\Phi_\nu(x) = \nu x^{\nu-1}$ , so that  $\lim_{\nu \rightarrow 0} \Phi_\nu \rightarrow \delta(x)$ , and therefore  $\lim_{\nu \rightarrow 0} P_\nu(t) = P(t)$ .

##### 4.2. Two-loop FAPT

In the two-loop approximation we have more complicated recurrence relation

$$-\frac{1}{n+\nu} \dot{\mathcal{F}}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1 \mathcal{F}_{n+2+\nu}[L] \quad (19)$$

with  $c_1$  being the corresponding coefficient in Eq. (2). In order to resum the series  $\mathcal{W}_\nu[L; t; \mathcal{F}]$  we need to introduce the “two-loop evolution” time

$$\tau_2(t) = t - c_1 \ln \left[ 1 + \frac{t}{c_1} \right]; \quad \dot{\tau}_2(t) = \frac{1}{1 + c_1/t}. \quad (20)$$

We obtained in [17] the following resummation recipe

$$\begin{aligned} \mathcal{W}_\nu[L, t; \mathcal{F}] &= \mathcal{F}_{\nu+1}[L] + \Delta(\nu) c_1 \dot{\tau}_2(t) \mathcal{F}_2[L_{t,0}] \\ &\quad - \dot{\tau}_2(t) \int_0^1 z^\nu \left[ t \dot{\mathcal{F}}_{\nu+1}[L_{t,z}] - \frac{c_1 \nu}{z} \mathcal{F}_{\nu+2}[L_{t,z}] \right] dz \end{aligned} \quad (21)$$

with  $L_{t,z} = L + \tau_2(tz) - \tau_2(t)$ ,  $L_{t,0} = L - \tau_2(t)$ , and  $\Delta(\nu)$  being a Kronecker delta symbol. Interesting to note here that it is possible to obtain an analogous, but more complicated recipe for the case when  $\mathcal{F}$  is the analytic image of the two-loop evolution factor  $a^\nu(1 + c_1 a)^{\nu_1}$ , see in [17] for more detail.

Table 2: Coefficients  $d_n$  for the Adler-function series with  $N_f = 4$ . The numbers in the square brackets denote the lower and the upper limits of the INNA estimates.

	PT coefficients	$d_1$	$d_2$	$d_3$	$d_4$	$d_5$
1	pQCD results with $N_f = 4$ [18, 19]	1	1.52	2.59	27.4	—
2 <sup>+</sup>	Model (26) with $c = 3.544$ , $\delta = 1.3252$	1	1.53	2.80	30.9	2088
2	Model (26) with $c = 3.553$ , $\delta = 1.3245$	1	1.52	2.60	27.3	2025
2 <sup>-</sup>	Model (26) with $c = 3.568$ , $\delta = 1.3238$	1	1.52	2.39	23.5	1969
3	“INNA” prediction of [17]	1	1.44	[3.5, 9.6]	[20.4, 48.1]	[674, 2786]

### 4.3. Three-loop FAPT

We describe here the recently obtained results on resummation in the three-loop FAPT. In this case the recurrence relation has three terms in the r.h.s.

$$-\frac{1}{n+\nu} \dot{\mathcal{F}}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1 \mathcal{F}_{n+2+\nu}[L] + c_2 \mathcal{F}_{n+3+\nu}[L] \quad (22)$$

with  $c_2$  being the corresponding coefficient in Eq. (2). We introduce the “three-loop evolution” time by

$$\begin{aligned} \tau_3(t) &= t + \frac{c_1^2 - 2c_2}{\Delta} \arctan \left[ \frac{t\Delta}{2c_2 + c_1 t} \right] \\ &\quad - \frac{c_1}{2} \ln \left[ 1 + \frac{c_1 + t}{c_2} t \right]; \quad (23) \\ \frac{d\tau_3(t)}{dt} &\equiv \dot{\tau}_3(t) = \frac{1}{1 + c_1/t + c_2/t^2}. \end{aligned}$$

Then our resummation recipe is

$$\begin{aligned} \mathcal{W}_\nu[L, t; \mathcal{F}] &= \mathcal{F}_{\nu+1}[L] + \Delta(\nu) c_2 \dot{\tau}_3(t) \mathcal{F}_3[L_{t,0}] \\ &+ t \mathcal{F}_{\nu+2}[L] - \dot{\tau}_3(t) \int_0^1 z^\nu \left[ t \dot{\mathcal{F}}_{\nu+1}[L_{t,z}] - \frac{c_2 \nu}{z} \mathcal{F}_{\nu+3}[L_{t,z}] \right. \\ &\quad \left. + z t^2 \dot{\mathcal{F}}_{\nu+2}[L_{t,z}] + (\nu+1) t \mathcal{F}_{\nu+2}[L_{t,z}] \right] dz \quad (24) \end{aligned}$$

with  $L_{t,z} = L + \tau_3(tz) - \tau_3(t)$  and  $L_{t,0} = L - \tau_3(t)$ .

## 5. Resummation for Adler function

Here we consider the power series of the vector correlator Adler function (labeled by the symbol V) [18, 19]

$$D_V[L] = 1 + \sum_{n \geq 1} d_n \left( \frac{\alpha_s[L]}{\pi} \right)^n. \quad (25)$$

Due to  $d_1 = 1$  coefficients  $\tilde{d}_n$  coincide with  $d_n$ . We suggested [17] the model for the generating function of the perturbative coefficients  $d_n$  (see 1st row in Table 2)

$$P_V(t) = \frac{\delta e^{-t/c\delta} - (t/c) e^{-t/c}}{c(\delta^2 - 1)}, \quad (26a)$$

which provides the following Lipatov-like coefficients

$$d_n^V = c^{n-1} \frac{\delta^{n+1} - n}{\delta^2 - 1} \Gamma(n). \quad (26b)$$

Our prediction  $d_4^V = 27.1$ , obtained with this generating function by fitting the two known coefficients  $d_2$  and  $d_3$  and using the model (26), is in a good agreement with the value 27.4, calculated in Ref. [18, 19]. Note that fitting procedure, taking into account the fourth-order coefficient  $d_4$ , produces the readjustment of the model parameters in (26) to the new values  $\{c = 3.5548, \delta = 1.32448\} \rightarrow \{c = 3.5526, \delta = 1.32453\}$ . The corresponding values of coefficients  $d_n^V$  are shown in the third row labelled by 2 in Table 2.

In order to understand how important are the exact values of the higher-order coefficients  $d_n$ , we employed our model (26) with two different sets of parameters  $c$  and  $\delta$ , shown in rows labelled by 2<sup>+</sup> and 2<sup>-</sup> in Table 2. One set, 2<sup>+</sup>, roughly speaking, enhances the exact values of the coefficients  $d_3$  and  $d_4$  by approximately +8% and +13%, correspondingly, while the other one, 2<sup>-</sup>, — reduces them in the same proportion. All coefficients of these models are inside the range of uncertainties determined in [17] using the Improved Naive Non-Abelization (INNA). Moreover, the difference between the analytic sums of the two models in the region corresponding to  $N_f = 4$  is indeed very small, reaching just a mere  $\pm 0.05\%$ . This gives an evident support for our model evaluation.

Now we are ready to estimate the relative errors,  $\Delta_N^V[L]$ , of the APT series<sup>2</sup> truncation at the  $N$ th term:

$$\mathcal{D}_N^V[L] = 1 + \sum_{n \geq 1}^N \frac{d_n}{\pi^n} \bar{\mathcal{A}}_n[L]; \quad (27a)$$

$$\Delta_N^V[L] = \frac{\mathcal{D}_\infty^V[L] - \mathcal{D}_N^V[L]}{\mathcal{D}_\infty^V[L]}. \quad (27b)$$

<sup>2</sup>Note that power series (25) has  $\nu = 0$  — for this reason we use here the APT approach.



Here  $\mathcal{D}_\infty^V[L]$  is the resummed APT result in the corresponding  $l$ -loop approximation, see Eqs. (18a), (21), and (24) with substitution  $\nu \rightarrow 0$ . In Fig. 1 we show these relative errors for  $N = 1, 2, 3$ , for the one- and two-loop cases (calculations for the three-loop case is not yet finished). The main result is in some sense surprising: The best order of truncation of the FAPT series in the region  $Q^2 = 2 - 20 \text{ GeV}^2$  is reached by employing the N<sup>2</sup>LO approximation, i.e., by keeping just the  $d_2$ -term.

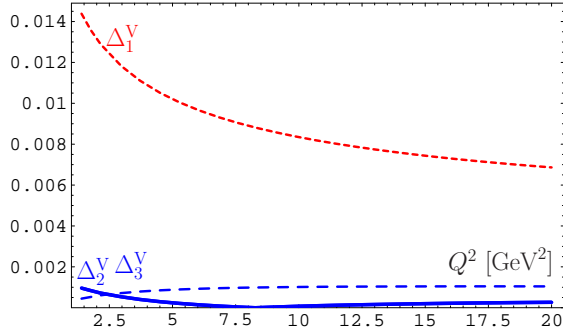


Figure 1: The relative errors  $\Delta_N^V(Q^2)$  evaluated for different values of  $N$ :  $N = 1$  (short-dashed red line),  $N = 2$  (solid blue line), and  $N = 3$  (dashed blue line) of the truncated APT given by Eq. (27b), in comparison with the exact result of the one- and two-loop resummation procedure represented by Eqs. (18) and (21).

We may also compare the numerical values for the resummed quantities, obtained in different loop approximations. We take for this comparison the following  $l$ -loop QCD scale parameters at  $N_f = 3$  flavors:  $\Lambda_3^{(l=1)} = 201 \text{ MeV}$ ,  $\Lambda_3^{(l=2)} = 379 \text{ MeV}$ , and  $\Lambda_3^{(l=3)} = 385 \text{ MeV}$ , which have been determined from the condition that the APT prediction for the ratio  $R_{e^+e^-}(s = m_Z^2)$  should coincide with the “experimental” value 1.03904, determined in [19]. We obtain the following values of the resummed Adler functions, shown in the table form for two values of  $Q^2$ , namely 3 and 2  $\text{GeV}^2$ :

Loop order $l$	$l = 1$	$l = 2$	$l = 3$
$\mathcal{D}_\infty^{V, (l)}(Q^2 = 3 \text{ GeV}^2)$	1.1129	1.1131	1.1164
$\mathcal{D}_\infty^{V, (l)}(Q^2 = 2 \text{ GeV}^2)$	1.1221	1.1223	1.1257

Strictly speaking, our model for coefficients is valid only for  $Q^2 \gtrsim m_c^2 = 2.44 \text{ GeV}^2$ , so that the first value  $Q^2 = 3 \text{ GeV}^2$  was selected to show the results in the legitimate  $N_f = 4$  domain. The second value,  $Q^2 = 2 \text{ GeV}^2$ , was selected for the comparison with the recent estimate in [20], where the value  $\mathcal{D}_\infty^V(Q_2^2) = 1.1217$  has been obtained in the two-loop approximation using the so-called generalized Pade summation method.

## 6. Resummation for $H^0 \rightarrow \bar{b}b$ Decay Width

Here we analyze the Higgs boson decay to a  $\bar{b}b$  pair. For its width we have

$$\Gamma(H \rightarrow b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_H \tilde{R}_s(M_H^2) \quad (28)$$

with  $\tilde{R}_s(M_H^2) \equiv m_b^2(M_H^2) R_s(M_H^2)$  and  $R_s(s)$  is the  $R$ -ratio for the scalar correlator, see for details in [12, 21]. In the one-loop FAPT this generates the following non-power expansion:

$$\tilde{R}_s[L] = 3 \hat{m}_{(1)}^2 \left\{ \mathfrak{A}_{\nu_0}^{\text{glob}}[L] + d_1^s \sum_{n \geq 1} \frac{\tilde{d}_n^s}{\pi^n} \mathfrak{A}_{n+\nu_0}^{\text{glob}}[L] \right\}, \quad (29)$$

where  $\hat{m}_{(1)} = 8.21 - 8.53 \text{ GeV}$  is the RG-invariant of the one-loop  $m_b(\mu^2)$  evolution  $m_b^2(Q^2) = \hat{m}_{(1)}^2 \alpha_s^{\nu_0}(Q^2)$  with  $\nu_0 = 2\gamma_0/b_0(5) = 1.04$  and  $\gamma_0$  is the quark-mass anomalous dimension (for a discussion — see in [17]).<sup>3</sup>

We take for the generating function  $P(t)$  the model of [26] with  $\{c = 2.4, \beta = -0.52\}$

$$P_s(t) = \frac{(t/c) + \beta}{c(1 + \beta)} e^{-t/c}. \quad (30a)$$

It provides the following Lipatov-like coefficients

$$\tilde{d}_n^s = c^{n-1} \frac{\Gamma(n+1) + \beta \Gamma(n)}{1 + \beta} \quad (30b)$$

which are in a very good agreement with  $\tilde{d}_n^s, n = 2, 3, 4$ , calculated in the QCD PT [21], the corresponding values of coefficients  $\tilde{d}_n^s$  are shown in the third row labelled by 2 in Table 3. In order to estimate the importance of the higher-order coefficients  $d_n$  exact values, we proceed along the same lines as in Sect. 5: We employ our model (30) with two different sets of parameters  $c$  and  $\beta$ , shown in rows labelled by 2<sup>+</sup> and 2<sup>−</sup> in Table 3. One set, 2<sup>+</sup>, enhances the exact values of the coefficients  $d_3$  and  $d_4$  by approximately +13% and +20%, correspondingly, while the other one, 2<sup>−</sup>, — reduces them in the same proportion. Obtained in this way difference between the analytic sums of the two models in the region corresponding to  $M_H = 80 - 170 \text{ GeV}$  is indeed very small, not more than 0.5%. Note here also that the model prediction for  $d_5$  is very close to the prediction obtained using the Principle of Minimal Sensitivity (PMS) [22], shown in the row with label 3 in Table 3.

<sup>3</sup>Different values of  $\hat{m}_{(1)}$  is related with two different extractions of the RG-effective  $b$ -quark mass,  $\overline{m}_b(\overline{m}_b^2)$ , which we have taken from two independent analyses. One value originates from Ref. [24] ( $\overline{m}_b(\overline{m}_b^2) = 4.35 \pm 0.07 \text{ GeV}$ ), while the other was derived in Ref. [25] yielding  $\overline{m}_b(\overline{m}_b^2) = 4.19 \pm 0.05 \text{ GeV}$ .

Table 3: Coefficients  $d_n$  for the Higgs-boson-decay width series with  $N_f = 5$ .

	PT coefficients	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
1	pQCD results with $N_f = 4$ [21]	1	7.42	62.3	620	—
2 <sup>+</sup>	Model (30) with $c = 2.43$ , $\delta = -0.52$	1	7.85	68.5	752	10120
2	Model (30) with $c = 2.62$ , $\delta = -0.50$	1	7.50	61.1	625	7826
2 <sup>-</sup>	Model (30) with $c = 2.25$ , $\delta = -0.51$	1	6.89	52.0	492	5707
3	“PMS” predictions of [21–23]	—	—	64.8	547	7782

After verification of our model quality we apply the FAPT resummation technique to estimate how good is FAPT in approximating the whole sum  $\tilde{\mathcal{R}}_s[L]$  in the range  $L \in [12.4, 13.5]$  which corresponds to the range  $M_H \in [100, 170]$  GeV<sup>2</sup> with  $\Lambda_{N_f=3}^{\text{QCD}} = 201$  MeV. Here we need to use our resummation recipes (18a) and (21) with substitution  $\nu \rightarrow \nu_0 = 1.04$ . Note that in the two-loop approximation the one-loop evolution factor  $a^\nu$  transforms into a more complicated expression,  $a^\nu(1 + c_1 a)^{\nu_1}$ . This produces additional numerical complications, but qualitatively results are the same. For this reason we show explicitly only one-loop formulas.

We analyze the accuracy of the truncated FAPT expressions

$$\tilde{\mathcal{R}}_s[L; N] = 3 \hat{m}_{(1)}^2 \left[ \mathcal{A}_{\nu_0}^{\text{glob}}[L] + d_1^s \sum_{n=1}^N \frac{\tilde{d}_n^s}{\pi^n} \mathcal{A}_{n+\nu_0}^{\text{glob}}[L] \right] \quad (31)$$

and compare them with the resummed FAPT result  $\tilde{\mathcal{R}}_s[L]$  in the corresponding  $l$ -loop approximation<sup>4</sup> using relative errors  $\Delta_N[L] = 1 - \tilde{\mathcal{R}}_s[L; N]/\tilde{\mathcal{R}}_s[L]$ . We estimate these errors for  $N = 2$ ,  $N = 3$ , and  $N = 4$  in the analyzed range of  $L \in [11, 13.8]$  and show that already  $\tilde{\mathcal{R}}_s[L; 2]$  gives accuracy of the order of 2.5%, whereas  $\tilde{\mathcal{R}}_s[L; 3]$  of the order of 1%. That means that there is no need to calculate further corrections: at the level of accuracy of 1% it is quite enough to take into account only coefficients up to  $d_3$ . This conclusion is stable with respect to the variation of parameters of the model  $P_s(t)$  and is in a complete agreement with Kataev–Kim conclusion [27].

In Fig. 2 we show the results for the decay width  $\Gamma_{H \rightarrow b\bar{b}}^\infty(M_H)$  in the resummed two-loop FAPT, in the window of the Higgs mass allowed by the LEP and Tevatron experiments. Comparing this outcome with the one-loop result, shown as the upper strip in Fig. 2, reveals a 5% reduction of the two-loop estimate. This reduction consists of two parts: one part ( $\approx +7\%$ ) comes from the difference in the mass  $\hat{m}$ , while the other

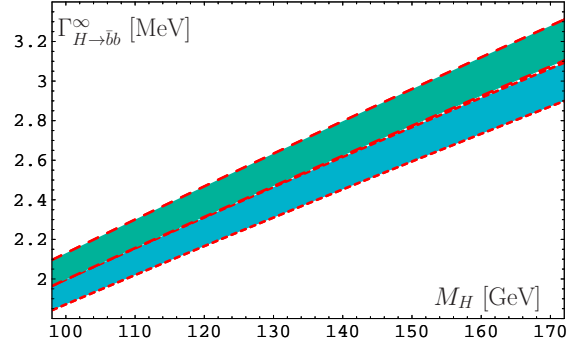


Figure 2: The two-loop width  $\Gamma_{H \rightarrow b\bar{b}}^\infty$  in the resummed FAPT as a function of the Higgs-boson mass  $M_H$ . The mass is varied in the interval  $\hat{m}_{(2)} = 8.22 \pm 0.13$  GeV according to the Penin–Steinhauser estimate [24]. The upper strip shows the corresponding one-loop result.

( $\approx -2\%$ ) is due to the difference in the values of  $R_s(M_H)$  in both approximations.

## 7. Conclusions

We conclude with the following.

APT provides natural way to Minkowski region for coupling and related quantities with weak loop dependence and practical scheme independence.

FAPT provides an effective tool to apply the APT approach for the renormalization-group improved perturbative amplitudes.

Both APT and FAPT produce finite resummed answers (now — up to the three-loop level) for perturbative quantities if we know the generating function  $P(t)$  for the PT coefficients.

Using quite simple model generating function  $P(t)$  for the Adler function  $\mathcal{D}(Q^2)$  we show that already at the N<sup>2</sup>LO an accuracy is of the order 0.1%, whereas for the Higgs boson decay  $H \rightarrow b\bar{b}$  at the N<sup>3</sup>LO is of the order of: 1% — due to the truncation error; 2% — due to the RG-invariant mass uncertainty.

<sup>4</sup>Here we show the results only for  $l = 1$  and  $l = 2$ : Calculations for  $l = 3$  are not yet finished.

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